Game Theory 101: The Complete Textbook

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Lesson 1.1: The Prisoner's Dilemma and Strict Dominance

The prisoner's dilemma is the oldest and most studied model in game theory, and its solution concept is also the simplest. As such, we will start with it.

Two thieves plan to rob an electronics store. As they approach the backdoor, the police arrest them for trespassing. The cops suspect that the pair planned to break in but lack the evidence to support such an accusation. They therefore require a confession to charge the suspects with the greater crime.

Having studied game theory in college, the interrogator throws them into the prisoner's dilemma. He privately sequesters both robbers and tells them the following:

We are currently charging you with trespassing, which calls for a one month jail sentence. I know you were planning on robbing the store, but right now I cannot prove it—I need your testimony. In exchange for your cooperation, I will dismiss your trespassing charge, and your partner will be charged to the fullest extent of the law: a twelve month jail sentence.

I am offering your partner the same deal. If both of you confess, your individual testimony is no longer as valuable, and your jail sentence will be eight months each.

If both criminals are self-interested and only care about minimizing their jail time, should they take the interrogator's deal?

1.1.1: Solving the Prisoner's Dilemma

The story contains a lot of information. Luckily, we can condense everything we need to know into a simple matrix:

	Keep Quiet	Confess
Keep Quiet	-1, -1	-12, 0
Confess	0, -12	-8, -8

We will use this type of game matrix regularly, so it is important to understand how to interpret it. There are two players in this game. The first player's strategies ("keep quiet" and "confess") are in the rows, and the second player's strategies are in the columns. The first player's payoffs are listed first for each outcome, and the second player's are listed second. For example, if the first player keeps quiet and the second player confesses, then the game ends in the top right set of payoffs, where the first player receives twelve months of jail time and the second player receives zero. Finally, as a matter of convention, we refer to the first player as a man and the second player as a woman; this will allow us to utilize pronouns like "he" and "she" instead of endlessly repeating "player 1" and "player 2."

Which strategy should each player choose? To see the answer, we must look at each move in isolation. Consider the game from player 1's perspective. Suppose he knew player 2 will keep quiet. How should he respond?

Let's focus on the important information in that context. Since player 1 is only thinking about his time in jail, we can block out player 2's payoffs with question marks:

	Keep Quiet
Keep Quiet	-1, ?
Confess	0, ?

Now we see that player 1 should confess. If he keeps quiet, he will spend one month in jail. But if he confesses, he walks away. Since he prefers less jail time to more jail time, he should confess in this situation.

Note that player 2's payoffs are completely irrelevant to player 1's decision in this context—if he knows that she will keep quiet, then he only needs to look at his own payoffs to decide which strategy to pick. Thus, the question marks could be any number at all, and player 1's optimal decision given player 2's move will remain the same.

On the other hand, suppose player 1 knew that player 2 will confess. What should he do? Again, the answer is easier to see if we only look at the relevant information:

	Confess
Keep Quiet	-12, ?
Confess	-8, ?

Confession wins a second time: confessing leads to eight months of jail time, whereas silence buys twelve. So player 1 would want to confess if player 2 confesses.

Putting these two pieces of information together, we reach an important conclusion—player 1 is better off confessing regardless of which strategy player 2 chooses! Thus, player 1 can effectively ignore whatever he thinks player 2 will do, since confessing gives him less jail time in either scenario.

Let's switch over to player 2's perspective. Suppose she knew that player 1 will keep quiet, even though we realize he should not. Here is her situation:

	Keep Quiet	Confess
Keep Quiet	?, -1	?, 0

As before, player 2 should confess, as she will shave a month off her jail sentence if she does so.

Finally, suppose she knew player 1 will confess. How should she respond?

	Keep Quiet	Confess
Confess	?, -12	?, -8

Unsurprisingly, she should confess and spend four fewer months in jail.

Once more, player 2 is better off confessing regardless of what player 1 does. Thus, we have reached a solution: both players confess, and both players spend eight months in jail. The justice system has triumphed, thanks to the interrogator's savviness.

This outcome perplexes a lot of people new to the field of game theory. Compare the <keep quiet, keep quiet> outcome to the <confess, confess> outcome:

	Keep Quiet	Confess
Keep Quiet	-1, -1	?, ?
Confess	?, ?	-8, -8

Looking at the game matrix, people see that the <keep quiet, keep quiet> outcome leaves both players better off than the <confess, confess> outcome. They then wonder why the players cannot coordinate on the mutually preferable outcome. But as we just saw, any promise to keep quiet is unsustainable. Player 1 *wants* player 2 to keep quiet so when he confesses he walks away free. The same goes for player 2. As a result, the <keep quiet, keep quiet> outcome is inherently unstable. Ultimately, the players end up in the inferior (but sustainable) <confess, confess> outcome.

1.1.2: The Meaning of the Numbers

Although there is a large branch of game theory devoted to the study of expected utility, we generally consider each player's payoffs as a ranking of his most preferred outcome to his least preferred outcome. In the prisoner's dilemma, we *assumed* that players only wanted to minimize their jail time. Game theory does *not* force players to have these preferences, as critics frequently claim. Instead, game theory analyzes what should happen given what players desire. So if players only want to minimize jail time, we could use the negative number of months spent in jail as their payoffs. This preserves their individual orderings over outcomes, as the most preferred outcome is worth 0, the least preferred outcome is -12, and everything else logically follows in between.

Interestingly, the cardinal values of the numbers are irrelevant to the outcome of the prisoner's dilemma. For example, suppose we changed the payoff matrix to this:

	Keep Quiet	Confess
Keep Quiet	3, 3	1, 4
Confess	4, 1	2, 2

Here, we have replaced the months of jail time with an ordering of most to least preferred outcomes, with 4 representing a player's most preferred outcome and 1 representing a player's least preferred outcome. In other words, player 1 would most like to reach the <confess, keep quiet> outcome, then the <keep quiet, keep quiet> outcome, then the <confess, confess> outcome, then the <keep quiet, confess> outcome.

Even with these changes, confess is still always better than keep quiet. To see this, suppose player 2 kept quiet:

	Keep Quiet
Keep Quiet	3, ?
Confess	4, ?

Player 1 should confess, since 4 is better than 3.

Likewise, suppose player 2 confessed:

	Confess
Keep Quiet	1, ?
Confess	2, ?

Then player 1 should still confess, as 2 is better than 1.

The same is true for player 2. First, suppose player 1 kept quiet:

	Keep Quiet	Confess
Keep Quiet	?, 3	?, 4

Player 2 ought to confess, since 4 is greater than 3.

Alternatively, if player 1 confessed:

	Keep Quiet	Confess
Confess	?, 1	?, 2

Player 2 should confess as well, as 2 is greater than 1. Thus, regardless of what the other player does, each player’s best strategy is to confess.

To be clear, this preference ordering exclusively over time spent in jail is just one way the players may interpret the situation. Suppose you and a friend were actually arrested and the interrogator offered you a similar deal. The results here do not generally tell you what to do in that situation, unless you and your friend only cared about jail time. Perhaps your friendship is strong, and both of you value it more than avoiding jail time. Since confessing might destroy friendship, you could prefer to keep quiet if your partner kept quiet, which changes the ranking of your outcomes. Your preferences here are perfectly rational. However, we do not yet have the tools to solve the corresponding game. We will reconsider these alternative sets of preferences in lesson 1.3.

Nevertheless, for the given payoffs, we have seen an example of *strict dominance*. We say that a strategy

X strictly dominates strategy Y for a player if strategy X provides a greater payoff for that player than strategy Y regardless of what the other players do. In this example, confess strictly dominated keep quiet for both players. Unsurprisingly, players never optimally select strictly dominated strategies—by definition, another strategy is *always* a better option regardless of what the other players do.

1.1.3: Applications of the Prisoner's Dilemma

The prisoner's dilemma has a number of applications. Let's use the game to explore optimal strategies in a number of different contexts.

First, consider two states considering whether to go to war. The military technology available to these countries gives the side that strikes first a large advantage in the fighting. In fact, the first-strike benefit is so great that each country would prefer attacking the other state even if its rival plays a peaceful strategy. However, because war destroys property and kills people, both prefer remaining at peace to simultaneously declaring war.

Using these preferences, we can draw up the following matrix:

	Defend	Attack
Defend	3, 3	1, 4
Attack	4, 1	2, 2

From this, we can see that the states most prefer attacking while the other one plays defensively. (This is due to the first-strike advantage.) Their next best outcome is to maintain the peace through mutual defensive strategies. After that, they prefer declaring war simultaneously. Each state's worst outcome is to choose defense while the other side acts as the aggressor.

We do not need to solve this game—we already have! This is the same game from the previous section, except we have exchanged the labels “keep quiet” with “defend” and “confess” with “attack.” Thus, we know that both states attack in this situation even though they both prefer the <defend, defend> outcome. The first-strike advantages trap the states in a prisoner’s dilemma that leads to war.

A similar problem exists with arms races. Imagine states must simultaneously choose whether to develop a new military technology. Constructing weapons is expensive but provides greater security against rival states. We can draw up another matrix for this scenario:

	Pass	Build
Pass	3, 3	1, 4
Build	4, 1	2, 2

Here, the states most prefer building while the other state passes. Following that, they prefer the <pass, pass> outcome to the <build, build> outcome; the states maintain the same relative military strength in both these outcomes, but they do not waste money on weaponry if they both pass. The worst possible outcome is for the other side to build while the original side passes. Again, we already know the solution to this game. Both sides engage in the arms race and build.

Now consider international trade. Many countries place tariffs (a tax) on imported goods to protect domestic industries even though this leads to higher prices overall.

We can use the prisoner’s dilemma to explain this phenomenon. A country can levy a tariff against another country’s goods or opt for no taxes. The best outcome for a country is to tax imports while not having the other country tax its exports. This allows the domestic industries to have an advantage at home and be competitive abroad, and the country also earns revenue from the tax itself. Free trade is the next best outcome, as it allows the lowest prices for each country's consumers. Mutual tariffs is the next best outcome, as they give each country an advantage at home but a disadvantage abroad; ultimately, this leads to higher prices than the free trade outcome. The worst possible outcome is to levy no taxes while the other country enforces a tariff, as domestic industries stand no chance against foreign rivals.

Let's toss that information into another matrix:

	No Tax	Tax
No Tax	3, 3	1, 4
Tax	4, 1	2, 2

We know this is a prisoner's dilemma and both sides will tariff each other's goods.

Finally, consider two rival firms considering whether to advertise their products. Would the firms ever want the government to pass a law forbidding advertisement? Surprisingly, if advertising campaigns only persuade a consumer to by a certain brand of product rather than the product in general, the answer is yes. If one side places ads and the other does not, the firm with the advertising campaign cuts into the other's share of the market. If they both advertise, the ads cancel each other out, but they still have to pay for the campaigns.

If we look at the corresponding matrix, we see another classic example of the prisoner's dilemma:

	No Ads	Ads
No Ads	3, 3	1, 4
Ads	4, 1	2, 2

Thus, both sides advertise to preempt the other side's campaign. The ads ultimately cancel each other out, and the firms end the game in a worse position than had they both not placed ads.

The Public Health Cigarette Smoking Act is a noteworthy application of the advertising game. In 1970, Richard Nixon signed the law, which removed cigarette ads from television. Tobacco companies actually *benefited* from this law in a perverse way—the law forced them to cooperating with each other. In terms of the game matrix, the law pushed them from the <2, 2> payoff to the mutually preferable <3, 3> payoff. The law simultaneously satisfied politicians, as it made targeting children more difficult for all tobacco companies.

These examples illustrate game theory's ability to draw parallels between seemingly dissimilar situations. We have seen models of prisoner confession, wars, arms races, taxation, and advertisements. Despite the range of examples, each had an underlying prisoner's dilemma mechanism. In this manner, game theory allows us to unify a wide-range of life decisions under a single, unified framework.

1.1.4: Deadlock

We can solve many games beyond the prisoner's dilemma using strict dominance alone. Take this game, for example:

	Left	Right
Up	3, 3	4, 1
Down	1, 4	2, 2

Here, up strictly dominates down for player 1 and left strictly dominates right for player 2. We can verify this using the same process as before. First, suppose player 2 chooses left:

	Left
Up	3, ?
Down	1, ?

We see that up is better than down, as 3 is greater than 1.

Repeating this for right, we focus on the following:

	Right
Up	4, ?
Down	2, ?

Once more, up is better than down, since 4 is greater than 2. So up is a better strategy than down regardless of what player 2 does.

Switching gears, suppose player 1 selected up. Then player 2 can focus on the following contingency:

	Left	Right
Up	?, 3	?, 1

Left is better than right in this case, as 3 is greater than 1.

Repeating this process a final time, player 2 now assumes player 1 will play down:

	Left	Right
Down	?, 4	?, 2

Left is still better than right, as 2 is greater than 1. Since left always beats right regardless of what player 1 does, left strictly dominates right, and therefore player 2 will play left. Thus, the outcome is <up, left>.

The technical name for this game is "deadlock"; both players are locked into their strictly dominant strategy and will never achieve their best outcome unless the other makes a mistake. However, unlike in the prisoner's dilemma, no alternative outcome exists that is simultaneously better for both players than the <up, left> solution. As such, this game is more intuitive but substantively less interesting.

1.1.5: Strict Dominance in Asymmetric Games

We can use strict dominance on games even when they are not as symmetric as the prisoner's dilemma or deadlock. Consider this one:

	Left	Right
Up	9, -2	3, 0
Down	8, 5	-1, 6

Unlike before, each player has a distinct set of payoffs. But if we run through the same process as before, we will see that <up, right> is the only reasonable solution.

Let's begin with player 1's choices. Suppose player 2 moved left. How should player 1 respond?

	Left
Up	9, ?
Down	8, ?

If he chooses up, he earns 9; if he picks down, he earns 8. Since 9 is greater than 8, player 1 should play up in response to left.

Now suppose player 2 chose right. Again, we need to find how player 1 should optimally respond.

	Right
Up	3, ?
Down	-1, ?

Up nets player 1 a payoff of 3, while down earns him -1. Since 3 is greater than -1, up is the better response to right. Thus, player 1 should play up regardless of player 2's strategy.

We know player 1's optimal strategy. All we need to do is repeat this process for player 2, and we will be done. Let's start with how player 2 should respond to up:

	Left	Right
Up	?, -2	?, 0

If player 2 chooses left, she earns -2; if she plays right, she earns 0. Since 0 is greater than -2, she should pick right in response to up.

Let's switch to player 2's response to down:

	Left	Right
Down	?, 5	?, 6

If player 2 selects left, she earns 5; if she chooses right, she earns 6. Since 6 is greater than 5, she should play right in response to down. Thus, regardless of player 1's choice, player 2 should optimally select right.

Therefore, the solution to this game is <up, right>. Player 1 ultimately earns 3, while player 2 earns 0.

Conclusion

Overall, strict dominance is a powerful tool in game theory. But while the concept is simple, applying it can be difficult. Even in matrix form, a game still has a lot of information. To successfully find dominated strategies, we must focus on one player's payoffs at a time. Above, we used question marks to isolate the relevant payoffs. When searching for strictly dominated strategies on your own, mentally block out the irrelevant payoffs and strategies in a similar manner.