A Note on Comparative Statics and Derivatives

The book and lecture materials point to derivatives as the best method to analyze comparative statics. This represents a problem for students who do not have a background in calculus. Why don't game theorists make life easier for everyone use simpler methods to calculate comparative statics instead?

Let's look at alternative strategies and see why the fail. One potential method is to solve the multiple versions game, tweaking the same parameter each time. For example, in [the penalty kick game](), the goalie's equilibrium probability of diving to the left equals x/(1+x), where x is between 0 and 1. While derivatives can tell us how this changes, plugging in multiple values of x can give some information.

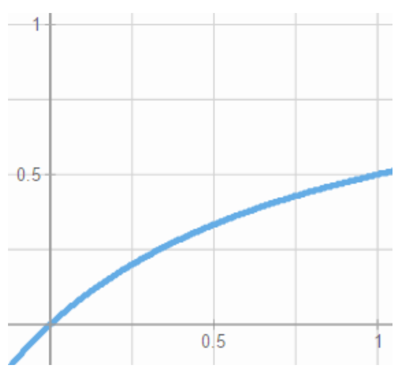
For example, when x = .25, the fraction becomes:

$$.25/(1+.25)$$
$$.25/1.25$$
$$1/5$$

And when x = .5, the fraction becomes:

$$.5/(1+.5)$$
$$.5/1.5$$
$$1/3$$

This suggests that increasing x increases the probability of diving to the left. Inspecting a graph reveals that this is correct:



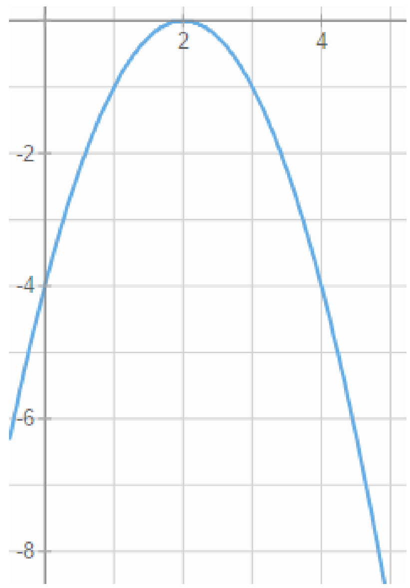
However, for some games, plugging in numbers in this manner and inspecting graphs does not work. For example, suppose that a player's equilibrium utility equals $-(z-2)^2$, where z > 0. You might check whether the equilibrium probability is increasing or decreasing by using z = 1 and z = 5. Here is the first case:

$$-(1-2)^2$$
$$-(-1)^2$$
$$-1$$

And the second case:

$$-(5-2)^2$$
$$-(3)^2$$
$$-9$$

Thus, you might conclude that increasing z decreases the player's utility. While it is true that moving y from 1 to 5 leads to a utility loss, this tells us nothing about the localized effect of changing z. To wit, take a look at the graph of the function:



The actual relationship is *non-monotonic*—i.e., the utility is increasing some portion of the time and decreasing some other portion of the time. Derivatives can reveal this information to us, which is why calculus is necessary to study game theory at a high level.

One may also wonder why we cannot just look at graphs like we have done here to sidestep the derivative problem. There are two issues with this. First, these graphs only give us a limited domain of values. It is possible, for example, that the equilibrium utility in the previous graph starts to increase again when z reaches 10^{20}. It would take a ton of zooming out to see that, and even then we would be could not conclude what happens as z grows even larger.

Second, the functions we have plotted here only had one exogenous variable. Suppose instead that a player's expected utility were $xy/(z-2)^2$ instead. With three variables, we cannot readily graph the results, which causes us to lose all visual cues.

Derivatives solve these problems, which is why we use them wherever we can. There is an alternative for those who do not know calculus or for when taking derivatives is especially messy. Recall once more that the goalie's equilibrium probability of diving left is $x/(1+x)$. Suppose that we want to show that this probability increases as x increases. One way to do this is to compare the equilibrium outcome under x and $x + \varepsilon$, where $\varepsilon > 0$. If we think of ε as being an especially small value, it can help us understand what happens to the function given a subtle change to the inputs.

Applying this to the penalty kick example, our goal is to show:

$$(x+\varepsilon)/(1+x+\varepsilon) > (x)/(1+x)$$

Applying some algebraic manipulation:

$$(1+x)(x+\varepsilon) > x(1+x+\varepsilon)$$
$$x + \varepsilon + x^2 + x\varepsilon > x + x^2 + x\varepsilon$$
$$\varepsilon > 0$$

This is true. Thus, adding any ε value to x increases the probability of diving left.

The ε method is a fine alternative even if it is not ideal.