Contents

1 Strategic Form Games

1.1 Dominance Problem #1

Use strict dominance and iterated elimination of strictly dominated strategies to find the solution to the following game, if such a solution exists:

	Left	Center	Right
Up	$6, 4$	$3, 5$	$9, 3$
Middle	$0, 1$	$4, 2$	$8, -1$
Down	$5, 0$	$2, -2$	$7, -1$

1.2 Dominance Problem #2

Use strict dominance and iterated elimination of strictly dominated strategies to find the solution to the following game, if such a solution exists:

	Left	LC	RC	Right
Up	$1, 7$	$-1, 6$	$6, 3$	$0, 5$
UM	$0, 3$	$-2, 5$	$6, 4$	$-4, 0$
DM	$0.5, 8$	$-2, 9$	$5, -7$	$-5, 0$
Down	$2, 3$	$-4, 2$	$-1, 1$	$3, 5$

1.3 Collective Action Problems

200 countries simultaneously choose whether to reform their domestic economies to produce cleaner energy. The cost for doing so 5. Fortunately, the gains are enormous: every country that shifts to clean energy adds 600 units to the collective welfare. However, clean air is a “public good”; it travels around the globe, so 600 units is split evenly among the countries. Not producing clean energy is free but does not provide any benefits.

a) Let n be the number of *other* countries that produce the good. What is a generic country’s expected utility for producing the good as well?

b) Holding n fixed from before, what is a generic country’s expected utility for not producing the good?

c) Using strict dominance, what is the solution to this game?

d) Using your answer for part (c), calculate each country’s expected payoff.

e) Now suppose that any country that produces has an additional gain of 10 units all to itself but only 400 units (instead of 600) are added to the collective welfare. Repeat parts (a) through (d) with this alternative assumption.

f) The benefit described in part (e) seems selfish—after all, those 10 units only go to the country that contributed. Why is it better to live in this world where selfish benefits exist regardless of whether you contribute to good?

1.4 A Beautiful Disaster

The film *A Beautiful Mind* had one job: explain game theory to the masses. They royally screwed up.

Consider the following scene from the film. John Nash and four of his closest friends are sitting at a bar. A blonde woman walks in. Simultaneously, Nash and his friends must decide whether to approach the blonde or a brunette.[1] There are enough brunettes in the room that approaching a brunette guarantees a successful date. However, each the men prefers going on a date with the blonde than one of the brunettes. Here’s the rub: if more than one man approaches the blonde, the blonde will feel like a piece of meat and shun all of their advances. This leaves any man that approached the blonde in a worse position than if he approached a brunette instead.

In the film, Nash proposes that they do not compete for the blonde. Instead, the solution is for each of them to approach a brunette. Explain why this is wrong—i.e., why Nash’s proposal is not a Nash equilibrium.[2]

[1] There is a clear feminist critique available for the rest of this problem. Please attribute it to the fact that this scene was supposed to take place in the 1940s.

[2] In a bit of unintended humor, one of Nash’s friends tells him “If this some way for you to get the blonde on your own, you can go to hell.”

1.5 Preventive War and Hidden (but Costly) Weapons Construction

States must worry that their rivals are secretly building weapons capable of great destruction. One possible solution is to declare *preventive war*, militarily defeat the other side, and ensure that such weapons (if they exist) will never be a problem. Both sides' strategies have significant drawbacks, though: war is costly to both states while building weapons is expensive to the rival.

With that in mind, consider the following interaction. State 1 must decide whether to *prevent* or *pass*. Meanwhile, state 2 must decide whether to *build* or *not build*. If state 1 passes and state 2 does not build, state 1 receives .8 and state .2 receives .2. However, if state 1 passes and state 2 builds, power shifts in state 2's favor. As such, state 1 receives .3 and state 2 receives .1. (The remaining .6 is lost to the weapons construction.) If state 1 prevents and state 2 does not build, state 1 receives .6 and state 2 receives .2. (The remaining .2 is lost in the costs of war.) Finally, if state 1 prevents and state 2 builds, state 1 receives .6 and state 2 receives -.4. (This time, .8 is lost through the costs of war plus the cost of weapons construction.)

a) Use the above information to construct and appropriately label a game matrix.

b) Use iterated elimination of strictly dominated strategies to find the solution to the game.

c) Note that state 1's worst outcome is to pass while state 2 builds. Why can state 1 trust state 2 in this case?

1.6 Preemptive War

Players 1 and 2 are in conflict over a strip of territory valued at 1. They first must decide whether to *bargain* or *fight*. If they both bargain, assume that player 1 will ultimately keep .6 of the territory and player 2 will keep .4. If they both fight, player 1 expects to win the war with probability .6 and player 2 expects to win the war with probability .4, but both will pay .1 in costs. If one bargains while the other fights, the fighter receives a first strike advantage: he will be .2 more likely to win the war, which also makes the other party .2 *less* likely to win. (The base probabilities of victory are the same as before. If any player starts a war, *both* states pay their costs.)

a) Draw a 2x2 game matrix. Label the strategies as *bargain* and *fight*. Calculate and fill in each player's payoff for the corresponding outcomes.

b) Use strict dominance to solve the game.

c) Is the outcome efficient? How does this game resemble a prisoner's dilemma?

1.7 Best Response Game #1

Find all pure strategy Nash equilibria of the following game by marking best responses:

	Left	LC	RC	Right
Up	$0, -7$	$0, 1$	$6, -3$	$0, 0$
UM	$-10, 3$	$4, 2$	$6, 4$	$-4, 7$
DM	$2.5, 3$	$-3, 0$	$1, -1$	$-2, 2$
Down	$4, 5$	$0, 0$	$-3, 2.5$	$2, 1$

1.8 Best Response Game #2

Find all pure strategy Nash equilibria of the following game by marking best responses:

	Left	LC	Center	RC	Right
Up	$0, -7$	$0, -7$	$0, -7$	$3, -7.5$	$0, -7$
UM	$0, -7$	$0, 1$	$6, 6$	$3, 3$	$0, -7$
Middle	$-1, -3$	$-1, -2$	$4, 4$	$-4, 7$	$3, 1$
DM	$2, 2$	$-4, -1$	$-1, -1$	$0, 0$	$0, 0$
Down	$3, 5$	$5, 6$	$-3, 2.5$	$2, -1$	$0, 0$

1.9 First Strike Advantages with Costly War

Previously, we looked at how war may be inevitable when conflict is cheap and first strike advantages exist. We now revisit the situation when war is more expensive.

Recall that players 1 and 2 are in conflict over a strip of territory valued at 1. They first must decide whether to *bargain* or *fight*. If they both bargain, assume that player 1 will ultimately keep .6 of the territory and player 2 will keep .4. If they both fight, player 1 expects to win the war with probability .6 and player 2 expects to win the war with probability .4, but both will pay .3 in costs. (Before, the costs of war were only .1 for each player.) If one bargains while the other fights, the fighter receives a first strike advantage: he will be .2 more likely to win the war, which also makes the other party .2 *less* likely to win. (The base probabilities of victory are the same as before. If any player starts a war, *both* states pay their costs.)

a) Draw a 2x2 game matrix. Label the strategies as *bargain* and *fight*. Calculate and fill in each player's payoff for the corresponding outcomes.

b) Find the game's Nash equilibria.

c) What does the game say about the rationality of fighting wars that no wants?

1.10 Hacking and Defending

State 1 employs hackers who can attack server A or server B. State 2 only has enough resources to adequately defend one of the servers from hacking. If the defending state correctly anticipates which server the hacker, both state receive 0. If the hacking state attacks A while the defending state protects B, the hacking state earns 5 while the defending state earns -5. If the hacking state attacks B while the defending state protects A, the hacking state earns 3 while the defending state earns -3. This is because server A contains more valuable information.

a) Represent this game in matrix form.

b) Find all pure strategy Nash equilibria.

c) Recall that Nash equilibria have a "no regret" policy—that is, after all players move and the outcome is revealed, no player individually regrets his or her strategy. How does this help make sense out of your answer to (b)?

d) Suppose the defending state hires a security advisor, who suggests that the state allocate its resources to protect server A. "After all, server A is the more valuable of the two. You wouldn't want to lose it." Is this sensible advice for the interaction described here? Why or why not?

e) Find all Nash equilibria.

1.11 Mixed Strategy Practice #1

Find the mixed strategy Nash equilibrium of the following game and calculate each player's equilibrium payoff.

	Left	Right
Up	$1, -1$	$0, 0$
Down	$-2, 3$	$5, -3$

1.12 Mixed Strategy Practice #2

Find the mixed strategy Nash equilibrium of the following game and calculate each player's equilibrium payoff.

	Left	Right
Up	$4, -1$	$-1, 7$
Down	$-5, 9$	$1.5, -2$

1.13 Mixed Strategy Practice #3

Find the mixed strategy Nash equilibrium of the following game and calculate each player's equilibrium payoff.

	Left	Center	Right
Up	$-3, 3$	$5, -4$	$7, -5$
Middle	$1, -3$	$-2, 2$	$0, 1$
Down	$0, 5$	$-4, -4$	$0, -6$

1.14 Mixed Strategy Practice #4

Find the mixed strategy Nash equilibrium of the following game and calculate each player's equilibrium payoff.

	Left	Center	Right
Up	$-8, 2$	$3, 1$	$0, 3$
Middle	$-5, 4$	$-3, -3$	$2, -9$
Down	$8, 0$	$-1, 4$	$3, 1$

1.15 System Protection

The federal government has two servers that are vulnerable to an attack, one in Los Angeles and the other in New York. Unfortunately, the government only has enough resources to reinforce one of the servers at the present time. The information on the Los Angeles server is more valuable than the one in New York. However, because more hardware personnel are headquartered on the East Coast, it is cheaper for the government to defend the New York server.

Consider the following two player game. Simultaneously, the government chooses to defend the Los Angeles server or the New York server, while a hacker selects which of the two to attack. If the hacker attacks the undefended server, she takes control of it. For the Los Angeles server, this gives her a payoff of 7 and the government a payoff of -7; for the New York server, this gives her a payoff of 5 and the government a payoff of -5. If the hacker picks the defended server, both receive a payoff of 0. In addition to those payoffs, the government also suffers a cost of -2 if it defends the Los Angeles server and -1 if it defends the New York server. (Thus, if the hacker attacks the Los Angeles server and the government defends the New York server, the government's overall payoff is -8.)

a) Write out the payoff matrix for this game.

b) Solve for its mixed strategy Nash equilibrium. What are each player's payoffs?

c) Now suppose defending either server costs -1. Write out the payoff matrix for the new game.

d) Solve for its mixed strategy Nash equilibrium. What are each player's payoffs?

e) Note that the government's cost for defending the Los Angeles server becomes cheaper in the second case and does not directly affect the hacker's payoff. How does this change alter the probability the government defends the Los Angeles server? What about the hacker's probability of attacking the Los Angeles server? What about each player's payoff?

1.16 Utility Sensitivity

Consider the following game:

	Left	Right
Up	$2, 4$	$0, 0$
Down	$0, 0$	$5, 2$

a) Find all Nash equilibria.

Now consider the following game:

	Left	Right
Up	$3, 4$	$0, 0$
Down	$0, 0$	$4, 1$

b) Find all Nash equilibria.

c) Do the pure strategy Nash equilibria change between the two games? What about the mixed strategy Nash equilibria?

d) Note that the two games do not differ in each player's rank ordering of preferences over outcomes. However, the *cardinality* (but not *ordinality*) of the payoffs has changed: player 1 now finds his middle outcome slightly better than before, whereas player 2 finds her middle outcome slightly worse. Use this information to help explain why the pure/mixed strategy Nash equilibria change or do not change. How general is this sensitivity or insensitivity? You may wish to tweak other games in this manner and solve them to test your theories.

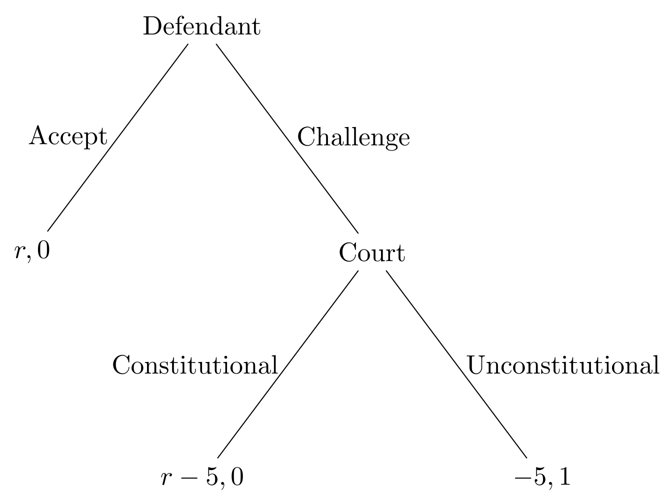
2 Extensive Form Games

2.1 Challenging Unconstitutional Laws

In the United States, all local, state, and federal laws must abide by the regulations of the United States Constitution. However, because the Constitution is not fully defined, many laws stand in a gray area between constitutionality and unconstitutionality. The Supreme Court has final say whether the Constitution permits any law, but it can only rule on constitutionality when someone harmed by an unconstitutional law challenges it in court. Overturning such a law correspondingly overrides any initial rulings against the defendant.

With that, consider the following game. A defendant chooses whether to challenge the law or not. If he does, the court rules on the constitutionality of the law. If he does not, he accepts the outcome of the trial. The court only wishes to make sure that laws follow the Constitution. The defendant cares about the outcome of the initial ruling but also finds challenging costly; win or lose the challenge, he suffers -5 in costs.

Suppose the law is unconstitutional. Then here is the game tree:



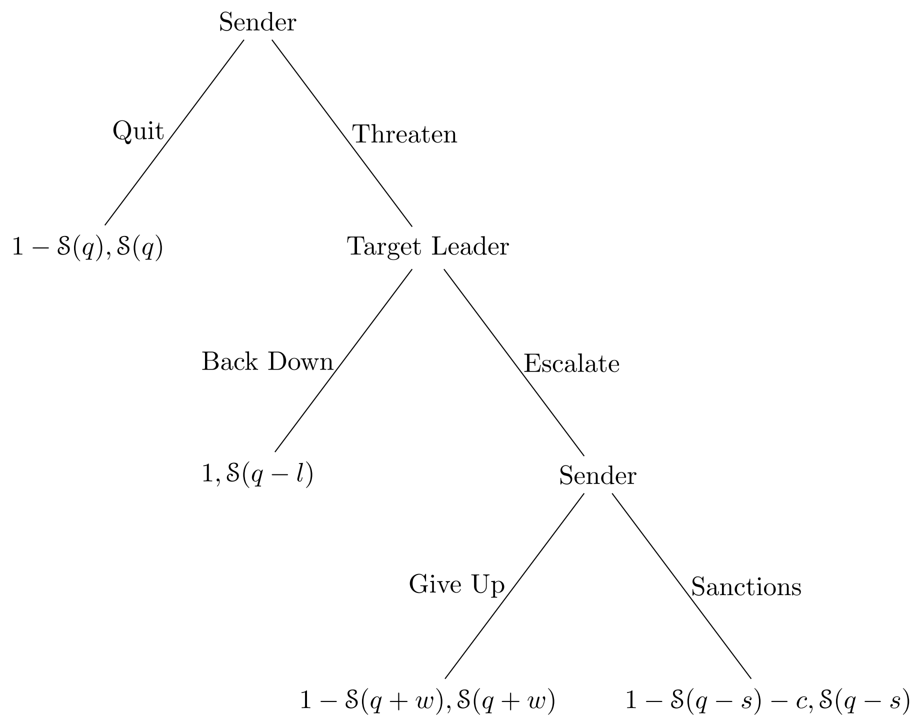
a) Let $r = -10$, representing the fact that the defendant was initially ruled guilty. Find the subgame perfect equilibrium.

b) Let $r = 0$, representing the fact that the defendant was initially ruled not guilty. Find the subgame perfect equilibrium.

c) The American court system routinely allows unquestionably guilty criminals to go free due to evidence obtained from unconstitutional laws. Given what you learned from parts (a) and (b), why is this a necessary evil? Put differently, what would happen in a world where the courts only allowed innocent people to challenge such unconstitutional laws?

2.2 Sanctions and Selection

Consider the following sanctions game between a sender of sanctions and its target:



The interaction is as follows. The sender state dislikes a policy that the target leader has implemented and wants change. It is considering sanctioning the leader to convince the leader to give up the policy or foment a domestic uprising against the leader.

Meanwhile, the leader simply wants to stay in office. As such, the target leader's payoffs are the probabilities he stays in office. Thus, if the sender quits without issuing a threat, the leader stays in power with probability $\mathcal{S}(q)$, with q reflecting the status **Q**uo. If the sender issues a threat and the target backs down, he stays in power with probability $\mathcal{S}(q-l)$, with l reflecting the leader's **L**oss in the crisis. If the sender issues a threat, the leader escalates, and the sender gives up, the leader stays in power with probability $\mathcal{S}(q+w)$, with w reflecting the leader's **W**in in the crisis. Finally, sanctions occur, the leader stays in power with probability $\mathcal{S}(q-s)$, with s reflecting the **S**anction's effectiveness.

The sender's payoffs are more complicated. If the leader backs down, the sender earns 1, reflecting how it achieves its aims. For the remaining outcomes, the sender can only obtain its goals if the leader loses power. As such, its payoffs in those cases are the probability the leader loses office. Additionally, the sender pays a cost c if it imposes sanctions so as to reflect the loss of trade efficiency.

a) Let $\mathcal{S}(q) = .8$, $\mathcal{S}(q-l) = .6$, $\mathcal{S}(q+w) = .9$, $\mathcal{S}(q-s) = .8$, and $c = .05$. (It may help to redraw the game tree with these payoffs explicitly written in.) What is the outcome of this game? Explain your answer.

b) Now suppose sanctions are more likely to cause the leader's removal, i.e., $\mathcal{S}(q-s) = .7$. Hold all other parameters at the same values as before. What is the outcome of this game? Explain your answer.

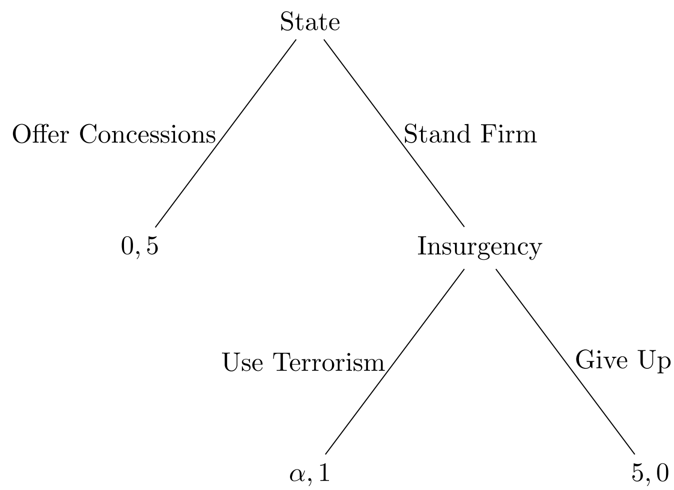
c) Now suppose sanctions are yet more likely to cause the leader's removal, i.e., $\mathcal{S}(q-s) = .5$. Hold all other parameters at the same values as before. What is the outcome of this game? Explain your answer.

d) Using the answers from above, explain why the sanctions we observe are not the most effective sanctions in principle. In answering this question, make sure to explain the strategic logic of why we fail to observe the most effective sanctions in practice.

2.3 Terrorism and Selection

Consider the following interaction between a state and an insurgent group. The state must decide whether to offer concessions to the insurgency or stand firm. The insurgency would most like to receive concessions, but these are costly for the state to give. If the state stands firm, the insurgency must decide whether to use terrorism to achieve its goals or give up. Terrorism may or may not ultimately be effective, but it will be costly to the state regardless.

Let α represent the government's tolerance for terror. Imagine that the preferences are as follows:



a) Suppose $\alpha = -1$. Find the subgame perfect equilibrium.

b) Now suppose $\alpha = 1$. Find the subgame perfect equilibrium of the new version of the game.

c) Note that the only difference between part (a) and part (b) is that the state finds terrorism slightly more tolerable in the second case. What do these models say about our ability to understand whether terrorism is effective at coercing concessions if we only look at instances where terrorism occurred?

2.4 Defending against Hackers

Safe Horizons Security Corporation has two servers, A and B, that a hacker is trying to infiltrate. The company has 80 units of effort to spend increasing the difficulty of hacking each of these servers. It first choose an allocation of those 80 units. Afterward, the hacker sees the chosen allocation and picks one of the servers to hack. If the hacker is successful in hacking the server she chooses, she receives 1. Otherwise, she receives 0. The company's payoffs are flipped; it receives 1 if the hacker fails and 0 if the hacker succeeds.

The chances of success depend on Safe Horizons' initial defense allocation. Let x be the number of units that the company invests in server A; thus, the number of units the company invests in server B is $80 - x$. Let the probability the hacker fails to hack server A equal $\frac{x+20}{100}$. Server B is less secure than server A. As such, the probability of a failed hack equal $\frac{80-x}{100}$.

a) Find the equilibrium allocation x.

b) Now suppose the company only has 30 units available to allocate to defense. Find the equilibrium allocation x under these conditions.

c) Using the answers to the above questions as guidance, substantively explain the goal of Safe Horizons' allocation decision.

3 Comparative Statics

3.1 Work or Shirk

Employers face a tradeoff between monitoring their employees and assuming that those employees are not shirking their duties. Meanwhile, lazy employees have incentive to socialize with their coworkers if they expect that no one will catch them.

Consider the following payoff matrix that represents these strategic concerns, with $p < 0$ representing a punishment payoff that an employee suffers when the employer catches her shirking:

	Work	Shirk
Monitor	$1, 1$	$0, p$
Trust	$2, 1$	$-1, 3$

a) Find the game's Nash equilibria.

b) An employer is considering making the punishment for shirking greater—that is, *decreasing* the value of p. Does the equilibrium probability that the employer monitor go up or down? What about the equilibrium probability the worker shirks?

c) Explain the intuition behind your answer for part (b).

3.2 Terrorists at an Airport

Consider an interaction between a terrorist and airport screeners. The terrorist can attempt an attack using his computer as a bomb or with liquid explosives. The screener only has time to search for one kind of contraband. If she guesses correctly, she will thwart the attack, and the terrorist will suffer a cost $c > 0$. If she is unsuccessful, the terrorist will gain a value $V_C > 0$ or $V_L > 0$ and the screener will lose that value, which depends on the vector of attack.

	Computer	Liquid
Computer	$0, -c$	$-V_C, V_C$
Liquid	$-V_L, V_L$	$0, -c$

a) Find the game's Nash equilibrium.

b) How does the terrorist's probability of choosing computer increase as V_C increases?

c) How does the screener's probability of choosing to search computers increase as V_C increases?

d) Suppose $V_C > V_L$. In equilibrium, which is more likely: a successful computer attack or a successful laptop attack?

3.3 Chicken

Consider the following game:

	Continue	Swerve
Continue	$-10, x$	$2, -2$
Swerve	$-2, 2$	$0, 0$

Suppose that $x < -2$. This gives the above game the form of "chicken" found in Lesson 1.6 of the textbook, except that player 2's disaster payoff has been generalized. Note that as x decreases, disaster looks worse and worse for player 2.

a) Find all Nash equilibria.

b) In the mixed strategy Nash equilibrium, how does the probability of player 2 continue change as a function of x? Explain substantively why this is.

c) In the mixed strategy Nash equilibrium, how does the probability that player 2 "wins" (i.e., player 1 serves and player 2 continues) change as a function of x? Explain substantively why this is.

3.4 The Judgment of Solomon

In 1 Kings 3:16-28, King Solomon of Israel is confronted with a dilemma. Two women approach him. Each individually claimed that they were the true mother of an infant boy and demanded remedy. Solomon declared that the obvious solution was to chop the boy in half so that the women could split him equally. One woman immediately relinquished her claim. Inferring that only the real mother would give up her claim so easily to avoid the death of the child, Solomon awarded her the baby. This fable is known as "The Judgment of Solomon" because Solomon (ostensibly) presented the women with the incentives that ultimately reveal the truth, allowing him to award the baby to the proper mother.

Let's formulate this as a game between the two mothers. Suppose that player 1 is the true mother; she values the child at v_T. Meanwhile, player 2 is the false mother; she values the child at v_F. Solomon makes his inference based on the fact that the true mother's value is greater than the false mother's, so let $v_T > v_F > 0$. Both claimants must simultaneously declare whether they are the mother. If only one declares she is not the mother, that woman receives the baby. If both say they are not the mother, Solomon awards the baby based on a coin flip. If both declare they are the mother, Solomon cuts the baby in half.

a) If the baby dies, suppose both claimants suffer their value for the child $(-v_T, -v_F)$. Draw the strategic form of the game. For what values of v_T and v_F is it an equilibrium for only the false mother to claim the baby?

b) In The Judgment of Solomon, the lying woman appears happy that the baby will be split in half. Thus, let's revise the payoff for both parties declaring motherhood to $(-v_T, v_F)$. That is, only the real mother feels pain if Solomon chops up the baby. For what values of v_T and v_F is it an equilibrium for only the false mother to claim the baby?

c) Using the same payoffs as in (b), now imagine that Solomon awards the baby to the true mother with probability $p \in (0, 1)$ if both retract their claim. For what values of v_T and v_F is it an equilibrium for only the false mother to claim the baby?

3.5 Judicial Nominees

Unlike regular pieces of legislation, senators face tremendous uncertainty when choosing whether to confirm judicial nominees. While presidents spend ample time having private conversations with potential nominees to understand their legal viewpoints, senators must always worry that an apparent moderate is really an extremist in disguise.

Consider the following worst case scenario in which the senate is completely in the dark regarding the president's decision. The senate most wants to confirm a moderate and least wants to confirm an extremist; rejection is its middle outcome. The president's payoffs are more complicated. He prefers having a moderate confirmed to rejection. Meanwhile, the value of attempting to nominate extremist depends on the level of backlash the president will eventually face when the public learns of the judicial nominee's true philosophy. Let $c > 0$ represent the cost he suffers in this outcome. Then the payoff matrix is as follows:

	Confirm	Reject
Moderate	$0, 1$	$-1, 0$
Extremist	$1 - c, -1$	$-1 - c, 0$

a) Find the game's Nash equilibria. Are they efficient?

b) Calculate the president's (player 1's) expected utility as a function of c. If you were the president and could choose any value for c, what would you pick?

c) Note that c is a cost that the president suffers. It can never directly add to his payoff. Is it therefore reasonable to assume that a president would want to minimize c whenever possible? Substantively explain why or why not.